Accelerated motion and special relativity transformations

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Abstract.

Transformation rules for coordinates, velocities and accelerations in accelerated reference frames are derived. A generalized approach of the special relativity is taken for a basis. A 7-dimensional space including projections of velocity vector as three additional coordinates to time and geometric coordinates is studied. Turns in pseudoplane $(\mathrm{d}t,\mathrm{d}v)$ of this 7-space describe accelerated motion of frame. In addition to the light velocity, the transformation formulas contain a fundamental constant which has a meaning of maximal acceleration. It is demonstrated that if a source of light moves with acceleration with respect to some reference frame, the light velocity in this frame is less than the constant c and depends on acceleration. The special relativity relation between energy, impulse, and mass gets changed for particle in accelerated motion. A generalized wave operator being invariant to the above transformations is introduced. The components of tensor and of potential of electromagnetic field get intermixing in transformation relations for accelerated frame.

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1. Introduction

At the present time the Special Relativity Theory (SRT) is widely used for the approximate description of motion in "non-fully-inertial" reference frames. Yet, there is a number of relativistic phenomena defying analysis in the framework of the SRT. In particular, the SRT allow to consider neither the process of formation (emission) of photon as particle moving with light velocity, nor its proper rotation. Moreover composition rules for accelerations, angular velocities, and for other kinematic parameters except velocities can in no way be found from the SRT. This problem has its origin in rough kinematic notions of light which form the basis of SRT.

The necessity of generalizing the SRT stems also from mathematical reasons. The matter is that a relativistic approach now-used for accelerated reference frames has the grave logical disadvantage. The additional algebraic relation between velocity and a turn angle, ψ , in pseudoplane (x,t)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c \tanh \psi$$

is introduced in the SRT. In particular, this relation allows to derive the relativistic composition rule for velocities. Yet, a similar algebraic relation is not considered for acceleration. Thus the uniformity of the description of different kinematic parameters is missing. For this reason, generalized relativistic relations for acceleration can escape our understanding and the composition rule for accelerations being similar to that of velocities can not be formulated.

The objective of this work is to develop a SRT generalization to accelerated motions from the principle of the uniform description of acceleration and of velocity. Adhering to this heuristic principle, one can achieve the essential progress in describing the kinematic properties of light. Further it will be shown that

- (i) light can move with acceleration;
- (ii) there is a maximal acceleration of light, A;
- (iii) the velocity, v, and acceleration, a, of light follow to the relation

$$c^2 - v^2 - \frac{c^2}{A^2}a^2 = 0;$$

(iv) the specified relation and the fundamental constants entered into it, the velocity c and the acceleration A, do not depend on velocity and acceleration of reference frame.

Note that the proposal of the existence of maximal acceleration is not radically new. It was first made by Caianiello in 1981 [3] in the context of a quantization model formulated in an eight-dimensional geometric phase space, with coordinates $x^{\alpha} = \{x^{i}, (\hbar/mc) u^{i}\}$, where x^{i} is the position four-vector, and $u^{i} = dx^{i}/ds$ is the relativistic four-velocity (i = 1, ..., 4).

2. Formulation of problem

Consider a body B in linear motion with respect to a frame K at velocity v and acceleration a. Let a frame K' move linear with respect to K at velocity v and acceleration a. Let the motion of B be characterized by velocity v' and acceleration a' with respect to K'.

In the Newtonian mechanics, the kinematics of linear uniform accelerated motion is described by the system of differential equations

$$dt = dt',$$

$$dx = \mathbf{v} dt' + dx',$$

$$dv = \mathbf{a} dt' + dv',$$

$$da = da' = 0.$$
(1)

We assume that the kinematics of linear uniform accelerated motion, in the

generalized SRT desired us, is described by a system

$$\begin{vmatrix} dt \\ dx \\ dv \end{vmatrix} = \mathbf{F} \begin{vmatrix} dt' \\ dx' \\ dv' \end{vmatrix},$$

$$da = da' = 0,$$
(2)

where the elements of transformation matrix \mathbf{F} are functions of velocity and of acceleration. For linear uniform motion, the specified system of differential equations should be reduced to the Lorentz transformations. Therefore

$$\mathbf{F}(\boldsymbol{v},0) = \begin{pmatrix} \frac{1}{\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}} & \frac{\boldsymbol{v}}{c^2\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}} & 0\\ \frac{\boldsymbol{v}}{\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}} & \frac{1}{\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}} & 0\\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover at the Newtonian limit it should be fulfilled

$$\mathbf{F}(oldsymbol{v},oldsymbol{a}) = \left(egin{array}{c|c} 1 & 0 & 0 \ \hline oldsymbol{v} & 1 & 0 \ \hline oldsymbol{a} & 0 & 1 \end{array}
ight) \,.$$

In a special case, the relations (2) should also reduce to the Rindler transformations (see, e.g., [1, 2])

$$t = \frac{c}{a} \sinh\left(\frac{at'}{c}\right), \qquad x = \frac{c^2}{a} \cosh\left(\frac{at'}{c}\right).$$
 (3)

The desired generalization of the SRT must meet to the specified requirements.

A common approach for the necessary generalizations is developed in Section 3 by using the standard SRT as an example. The accelerated motion kinematics, including composition rules for velocities and accelerations and transformations of kinematic variables, is considered in Section 4. In particular, the reduction to the Rindler transformations is produced. Section 4 contains, in addition, transformations of potential and of tensor of electromagnetic field for accelerated frames. A generalization of dynamic variables (energy, impulse, force) is produced in Section 5 where the standard wave equation is also modified to describe accelerated motion of wave. The conclusions are presented in Section 6.

3. Lorentz transformations: a generalized approach

In this Section the Lorentz transformations will be derived by a new original method which will be important for further analysis.

An interval square used in the SRT can be represented as

$$(ds)^2 = c^2 (dt)^2 - (dx)^2$$
.

The variables are changed conveniently by:

$$dx^1 = dx, \qquad dx^4 = c dt.$$

Using the new variables we have the interval square in form

$$(ds)^2 = (dx^4)^2 - (dx^1)^2$$
.

The linear transformation preserving the interval square

$$||\mathrm{d}x|| = \mathbf{U}\,||\mathrm{d}x'||\tag{4}$$

is a turn in (dx^1, dx^4) pseudoeuclidean plane [4]. From the transformation matrix

$$\mathbf{U} \equiv \mathbf{V} = \begin{pmatrix} \frac{\cosh \Psi & \sinh \Psi}{\sinh \Psi & \cosh \Psi} \end{pmatrix}$$

follows

$$dx^{4} = \cosh \Psi (dx^{4})' + \sinh \Psi (dx^{1})',$$

$$dx^{1} = \sinh \Psi (dx^{4})' + \cosh \Psi (dx^{1})',$$
(5)

where Ψ is the turn angle.

To find a relation between the angle Ψ and the velocity x_4^1 we consider the variation of coordinate differentials as function of the angle variation:

$$\delta ||\mathrm{d}x|| = \delta \mathbf{U} ||\mathrm{d}x'||$$
.

Taking into account that

$$||\mathrm{d}x'|| = \mathbf{U}^{-1} ||\mathrm{d}x||$$

we can set

$$\delta||\mathbf{d}x|| = (\delta \mathbf{U} \mathbf{U}^{-1}) ||\mathbf{d}x||. \tag{6}$$

In our case this relation is reduced to

$$\delta dx^4 = \delta \Psi dx^1,$$

$$\delta dx^1 = \delta \Psi dx^4.$$
(7)

Now let us consider the differential

$$\delta \frac{\mathrm{d}x^1}{\mathrm{d}x^4} = \frac{\delta \mathrm{d}x^1}{\mathrm{d}x^4} - \mathrm{d}x^1 \frac{\delta \mathrm{d}x^4}{(\mathrm{d}x^4)^2} \,. \tag{8}$$

Using (7) we find

$$\delta \frac{\mathrm{d}x^1}{\mathrm{d}x^4} = \delta \Psi \left[1 - \left(\frac{\mathrm{d}x^1}{\mathrm{d}x^4} \right)^2 \right] . \tag{9}$$

The relations (5) and (9) describe the Lorentz transformations and the SRT composition rule for velocities.

The principal positions of the mathematical approach used above must be emphasized:

- (i) A vector space is built on the differentials of function x(t) and of argument t.
- (ii) The linear transformations preserving the interval square of this space are considered.

- (iii) A correspondence exists between the said linear transformations and the derivative v = dx/dt.
- (iv) The rule of composition of the linear transformations corresponds to the composition rule in the space of derivatives (i.e. the velocity composition rule).

We assume that this approach has an invariant meaning and can be applied to the function v(t). This point of view allows to obtain transformations for accelerated frame and the composition rule for accelerations in according with the special relativity principles.

4. Transformations for accelerated frames

In addition to the function x(t) we consider the function v(t) = dx/dt. Correspondingly to the previous considerations we construct a vector space on differentials dx, dt, dv and set the interval square in the following form

$$(ds)^{2} = c^{2} (dt)^{2} - (dx)^{2} - T^{2} (dv)^{2}.$$

Here the constant T adjusts the dimensionality of dv to the dimensionality of interval and has the dimensionality of time. Setting the constant

$$L = cT$$

with the dimensionality of length and dividing the interval square by L^2 we obtain

$$(d\sigma)^2 \equiv \frac{(\mathrm{d}s)^2}{L^2} = \frac{(\mathrm{d}t)^2}{T^2} - \frac{(\mathrm{d}x)^2}{L^2} - \frac{(\mathrm{d}v)^2}{c^2}.$$

In this expression all components are dimensionless. Let us change variables

$$x^{1} = \frac{x}{L}, \qquad x^{4} = \frac{t}{T}, \qquad x_{4}^{1} = \frac{v}{c} = \frac{\mathrm{d}x^{1}}{\mathrm{d}x^{4}}.$$

Then the interval square is

$$(d\sigma)^2 = (dx^4)^2 - (dx^1)^2 - (dx_4^1)^2.$$
(10)

Further we shall make considerations about a sign of square of velocity differential in the interval square. One can define the dimensionless interval square through covariant and contravariant vector coordinates as

$$-(d\sigma)^2 = dx^4 dx_4 + dx^1 dx_1 + dx_4^1 dx_4^1.$$

Here all addends on the right have a positive sign. If we use by relations

$$dx_1 = g_{11} dx^1 = dx^1$$
, $dx_4 = g_{44} dx^4 = -dx^4$, $dx_1^4 = dx_1^4$,

where it is taken into account that the turn matrix in pseudoeuclidean plane does not change sign by transposition, we get (10). Thus the coordinate x_4^1 is *spacelike*.

Let us introduce the following notations for dimensionless motion parameters: x^1 , x^4 , x^4 , x^1 , x^4 , x^1 as coordinate, time, velocity and acceleration of body B with respect to frame K; $(x^1)'$, $(x^4)'$, $(x^4)'$, $(x^1)'$, $(x^1)'$ as that with respect to K'; and \mathbf{x}^1 , \mathbf{x}^4 , \mathbf{x}^1 , \mathbf{x}^1 as that for frame K' with respect to K.

A turn in the space of differentials

$$||\mathrm{d}x|| = \mathbf{U} ||\mathrm{d}x'||$$

preserves the interval square. The turns in pseudoplanes (dx^1, dx^4) and (dx^4, dx_4^1) :

$$\mathbf{V} = \begin{pmatrix} \frac{\cosh \Psi & \sinh \Psi & 0}{\sinh \Psi & \cosh \Psi & 0} \\ \hline 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} \frac{\cosh \Phi & 0 & \sinh \Phi}{0} \\ \hline 0 & 1 & 0 \\ \hline \sinh \Phi & 0 & \cosh \Phi \end{pmatrix}$$

describe uniform velocity and accelerated motions of frame, respectively. Since turns are in general non-commuting, a motion with $\mathbf{U} = \mathbf{V} \mathbf{A}$ must be distinguished from a motion with $\mathbf{U} = \mathbf{A} \mathbf{V}$.

4.1. **V A**-motion

Consider turns described by matrix U = V A. The relations (4) and (6) take form:

$$dx^{4} = \cosh \Psi \cosh \Phi (dx^{4})' + \sinh \Psi (dx^{1})' + \cosh \Psi \sinh \Phi (dx_{4}^{1})',$$

$$dx^{1} = \sinh \Psi \cosh \Phi (dx^{4})' + \cosh \Psi (dx^{1})' + \sinh \Psi \sinh \Phi (dx_{4}^{1})', \qquad (11)$$

$$dx_{4}^{1} = \sinh \Phi (dx^{4})' + \cosh \Phi (dx_{4}^{1})',$$

and

$$\delta dx^{4} = \delta \Psi dx^{1} + \cosh \Psi \delta \Phi dx_{4}^{1},$$

$$\delta dx^{1} = \delta \Psi dx^{4} + \sinh \Psi \delta \Phi dx_{4}^{1},$$

$$\delta dx_{4}^{1} = \cosh \Psi \delta \Phi dx^{4} - \sinh \Psi \delta \Phi dx^{1}.$$
(12)

To find a relation connecting the angles Ψ and Φ with the velocity x_4^1 we consider the differential δx_4^1 . Using (8) and (12) we obtain

$$\delta x_4^1 = \left[1 - (x_4^1)^2\right] \, \delta \Psi + \left(\sinh \Psi - x_4^1 \cosh \Psi\right) x_{44}^1 \, \delta \Phi \,. \tag{13}$$

To find a relation connecting the angles Ψ and Φ with the acceleration x_{44}^1 we consider the differential

$$\delta \frac{\mathrm{d}x_4^1}{\mathrm{d}x^4} = \frac{\delta \mathrm{d}x_4^1}{\mathrm{d}x^4} - \mathrm{d}x_4^1 \frac{\delta \mathrm{d}x^4}{(\mathrm{d}x^4)^2}.$$

Using (12) we obtain

$$\delta x_{44}^1 = -x_{44}^1 x_4^1 \delta \Psi + \left\{ \cosh \Psi \left[1 - (x_{44}^1)^2 \right] - \sinh \Psi x_4^1 \right\} \delta \Phi. \tag{14}$$

The relations (11), (13) and (14) describe the coordinate transformations for the accelerated motion of body B and of frame K' as well as the composition rules for velocities and accelerations.

Further we shall find a solution of equations (13) and (14), when the velocity of body B is zero with respect to K'. In this case $v = \mathbf{v}$.

4.1.1. Composition rule for accelerations From (13) under initial condition ($\Psi = 0$, $x_4^1 = 0$) follows

$$x_4^1 = \boldsymbol{x}_4^1 = \tanh \Psi. \tag{15}$$

If the velocity $x_4^1=0$ and $\Psi=0$ then (14) gets reduced to

$$\delta x_{44}^1 = \left[1 - (x_{44}^1)^2\right] \delta \Phi.$$

The solution of this equation is

$$x_{44}^1 = \tanh(\Phi + \phi') = \tanh\phi, \tag{16}$$

where ϕ' is integration constant and notation $\phi = \Phi + \phi'$ is used. The integration constant ϕ' can be found from the following considerations. Let us consider that $\phi' = 0$ corresponds a' = 0, and $\Phi = 0$ corresponds a = 0. Thus

$$\mathbf{x}_{44}^1 = \tanh \Phi$$
, $(x_{44}^1)' = \tanh \phi'$.

Hereof and from (16) the composition rule for accelerations follows

$$x_{44}^{1} = \frac{\boldsymbol{x}_{44}^{1} + (x_{44}^{1})'}{1 + \boldsymbol{x}_{44}^{1} (x_{44}^{1})'}.$$

Changing for dimensional acceleration in accordance with

$$x_{44}^1 = \frac{\mathrm{d}x_4^1}{\mathrm{d}x^4} = \frac{T}{c} \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{a}{A},$$

where the constant with dimensionality of acceleration

$$A = \frac{c}{T}$$

was introduced, we obtain

$$a = \frac{\boldsymbol{a} + a'}{1 + \frac{\boldsymbol{a} \cdot a'}{A^2}} \,.$$

Thus the resulting acceleration is always less than or equal to A. Note that the above composition rule for accelerations was derived by Scarpetta within the framework of Caianiello's model [5].

At non-zero velocity x_4^1 with using (15) the equation (14) yields

$$x_{44}^{1} = \sqrt{1 - (\boldsymbol{x}_{4}^{1})^{2}} \tanh(\Phi + \phi').$$
 (17)

Because of correspondence between ϕ' and a', Φ and a we have

$$\mathbf{x}_{44}^{1} = \sqrt{1 - (\mathbf{x}_{4}^{1})^{2}} \tanh \Phi \tag{18}$$

and $(x_{44}^1)' = \tanh \phi'$. In the last relation it is taken into account that $(x_4^1)' = 0$. Hereof and from (17) the general composition rule for accelerations follows

$$a = \sqrt{1 - \frac{v^2}{c^2}} \frac{a + \sqrt{1 - \frac{v^2}{c^2}} a'}{\sqrt{1 - \frac{v^2}{c^2}} + \frac{a a'}{A^2}}.$$

4.1.2. Transformation of coordinate differentials From (15) and (18) follows

$$\cosh \Psi = \frac{1}{\sqrt{1 - (\boldsymbol{x}_{4}^{1})^{2}}}, \quad \sinh \Psi = \frac{\boldsymbol{x}_{4}^{1}}{\sqrt{1 - (\boldsymbol{x}_{4}^{1})^{2}}}, \\
\cosh \Phi = \frac{\sqrt{1 - (\boldsymbol{x}_{4}^{1})^{2}}}{\sqrt{1 - (\boldsymbol{x}_{4}^{1})^{2} - (\boldsymbol{x}_{44}^{1})^{2}}}, \quad \sinh \Phi = \frac{\boldsymbol{x}_{44}^{1}}{\sqrt{1 - (\boldsymbol{x}_{4}^{1})^{2} - (\boldsymbol{x}_{44}^{1})^{2}}}.$$

Substituting the above expressions in (11) we obtain the transformations of coordinate differentials:

$$dx^{4} = \frac{1}{\sqrt{1 - (x_{4}^{1})^{2} - (x_{44}^{1})^{2}}} (dx^{4})' + \frac{x_{4}^{1}}{\sqrt{1 - (x_{4}^{1})^{2}}} (dx^{1})' + \frac{1}{\sqrt{1 - (x_{4}^{1})^{2}}} \frac{x_{44}^{1}}{\sqrt{1 - (x_{4}^{1})^{2} - (x_{44}^{1})^{2}}} (dx^{4})',$$

$$dx^{1} = \frac{x_{4}^{1}}{\sqrt{1 - (x_{4}^{1})^{2} - (x_{44}^{1})^{2}}} (dx^{4})' + \frac{1}{\sqrt{1 - (x_{4}^{1})^{2}}} (dx^{1})' + \frac{x_{4}^{1}}{\sqrt{1 - (x_{4}^{1})^{2}}} \frac{x_{44}^{1}}{\sqrt{1 - (x_{4}^{1})^{2} - (x_{44}^{1})^{2}}} (dx^{4})', \qquad (19)$$

$$dx_{4}^{1} = \frac{x_{44}^{1}}{\sqrt{1 - (x_{4}^{1})^{2} - (x_{44}^{1})^{2}}} (dx^{4})' + \frac{\sqrt{1 - (x_{4}^{1})^{2}}}{\sqrt{1 - (x_{4}^{1})^{2} - (x_{44}^{1})^{2}}} (dx^{4})'.$$

Thus we have found a concrete form of the transformations (2). By using dimensional values for $\mathbf{v} \ll c$ and $\mathbf{a} \ll A$ the transformations are reduced to

$$dt = dt' + \frac{1}{c^2} \boldsymbol{v} dx' + \frac{1}{A^2} \boldsymbol{a} dv',$$

$$dx = \boldsymbol{v} dt' + dx' + \frac{1}{A^2} \boldsymbol{a} \boldsymbol{v} dv',$$

$$dv = \boldsymbol{a} dt' + dv'.$$

At the Newtonian limit $(c \to \infty \text{ and } A \to \infty)$ we obtain the system of the differential equations (1).

4.2. A V-motion

We now consider the case when turns are described by matrix $\mathbf{U} = \mathbf{A} \mathbf{V}$. The coordinate transformations and the composition rules for velocities and accelerations can be obtained, much as it was made in the previous Section. We give results for a' = 0.

The velocity composition rule has form

$$v = \sqrt{1 - \frac{a^2}{A^2}} \frac{v + \sqrt{1 - \frac{a^2}{A^2}} v'}{\sqrt{1 - \frac{a^2}{A^2} + \frac{v v'}{c^2}}}.$$

If the body B is light source, the light speed with respect to K is calculated by

$$v = c\sqrt{1 - \frac{\boldsymbol{a}^2}{A^2}}.$$

Therefore the maximal velocity of accelerated motion is less than c.

The transformations of coordinate differentials have form

$$dx^{4} = \frac{1}{\sqrt{1 - (x_{4}^{1})^{2} - (x_{44}^{1})^{2}}} (dx^{4})' + \frac{x_{4}^{1}}{\sqrt{1 - (x_{4}^{1})^{2} - (x_{44}^{1})^{2}}} \frac{1}{\sqrt{1 - (x_{44}^{1})^{2}}} (dx^{1})' + \frac{x_{44}^{1}}{\sqrt{1 - (x_{44}^{1})^{2}}} (dx^{4})',$$

$$dx^{1} = \frac{x_{4}^{1}}{\sqrt{1 - (x_{44}^{1})^{2} - (x_{44}^{1})^{2}}} (dx^{4})' + \frac{\sqrt{1 - (x_{44}^{1})^{2}}}{\sqrt{1 - (x_{44}^{1})^{2} - (x_{44}^{1})^{2}}} (dx^{1})',$$

$$dx^{1}_{4} = \frac{x_{44}^{1}}{\sqrt{1 - (x_{44}^{1})^{2} - (x_{44}^{1})^{2}}} (dx^{4})' + \frac{x_{4}^{1}}{\sqrt{1 - (x_{44}^{1})^{2} - (x_{44}^{1})^{2}}} \frac{x_{44}^{1}}{\sqrt{1 - (x_{44}^{1})^{2}}} (dx^{1})' + \frac{1}{\sqrt{1 - (x_{44}^{1})^{2}}} (dx^{1})'.$$

$$(20)$$

By using dimensional values for $\mathbf{v} \ll c$ and $\mathbf{a} \ll A$ these transformations are reduced to

$$dt = dt' + \frac{1}{c^2} \mathbf{v} dx' + \frac{1}{A^2} \mathbf{a} dv',$$

$$dx = \mathbf{v} dt' + dx',$$

$$dv = \mathbf{a} dt' + \frac{1}{c^2} \mathbf{v} \mathbf{a} dx' + dv'.$$

Hereof the possibility of passing to the Newtonian kinematics follows also.

4.3. The special case: Rindler transformations

Let us derive the Rindler transformations, for example, from $\mathbf{V}\mathbf{A}$ -motion transformations. Consider the relations (11) when

$$(dx^1)' = 0, \quad (dx_4^1)' = 0.$$
 (21)

In this case $x_4^1 = \boldsymbol{x}_4^1, \, x_{44}^1 = \boldsymbol{x}_{44}^1$ and

$$dx^{4} = \cosh \Psi \cosh \Phi (dx^{4})',$$

$$dx^{1} = \sinh \Psi \cosh \Phi (dx^{4})',$$

$$dx_{4}^{1} = \sinh \Phi (dx^{4})'.$$
(22)

Here the angles Ψ and Φ are related with velocity and acceleration of frame K' by (15) and (18). Let these angles be small and

$$\Psi \approx x_4^1$$
, $\sinh \Phi \approx \tanh \Phi \approx x_{44}^1$, $\cosh \Phi \approx 1$.

Then the transformations (22) are reduced to

$$dx^{4} = \cosh(x_{4}^{1}) (dx^{4})',$$

$$dx^{1} = \sinh(x_{4}^{1}) (dx^{4})',$$

$$dx_{4}^{1} = x_{44}^{1} (dx^{4})'.$$

If we integrate these equations under the constancy of acceleration x_{44}^1 and assume integration constants are zeros, we obtain

$$x^4 = \frac{1}{x_{44}^1} \sinh[x_{44}^1(x^4)'], \qquad x^1 = \frac{1}{x_{44}^1} \cosh[x_{44}^1(x^4)'].$$

Hereof the Rindler transformations (3) follow. Note that the time t' is proper time $\tau = s/c$ in the Rindler transformations. Really, from (21) follows

$$(\mathrm{d}s)^2 = c^2 (\mathrm{d}t')^2.$$

4.4. Transformations for potential and tensor of electromagnetic field

Transformation formalism for potential and tensor of electromagnetic field follows from (19) and (20) with due regard that potential, \mathcal{A}^i , gets transformed similar to $\mathrm{d}x^i$, tensor components, F_4^b , get transformed similar to velocity $\mathrm{d}x_4^b$, and sets of components $\{F_2^4, F_2^1\} = \{-\mathrm{E}^2, B^3\}, \{F_3^4, F_3^1\} = \{-\mathrm{E}^3, -B^2\}$ get transformed similar to $\{\mathrm{d}x^4, \mathrm{d}x^1\}$ components. Because the transformations (19) and (20) do not affect on components $\mathrm{d}x^2$ and $\mathrm{d}x^3$ then components \mathcal{A}^2 , \mathcal{A}^3 and $F_3^2 = B^1$ remain constant.

In particular for A V-motion, the transformations have form

$$\varphi = \frac{1}{\sqrt{1 - (x_{4}^{1})^{2} - (x_{44}^{1})^{2}}} (\varphi)' + \frac{x_{4}^{1}}{\sqrt{1 - (x_{4}^{1})^{2} - (x_{44}^{1})^{2}}} \frac{1}{\sqrt{1 - (x_{44}^{1})^{2}}} (A^{1})' + \frac{x_{44}^{1}}{\sqrt{1 - (x_{44}^{1})^{2}}} (E^{1})',$$

$$A^{1} = \frac{x_{4}^{1}}{\sqrt{1 - (x_{4}^{1})^{2} - (x_{44}^{1})^{2}}} (\varphi)' + \frac{\sqrt{1 - (x_{44}^{1})^{2}}}{\sqrt{1 - (x_{44}^{1})^{2} - (x_{44}^{1})^{2}}} (A^{1})',$$

$$E^{1} = \frac{x_{44}^{1}}{\sqrt{1 - (x_{4}^{1})^{2} - (x_{44}^{1})^{2}}} (\varphi)' + \frac{x_{4}^{1}}{\sqrt{1 - (x_{44}^{1})^{2} - (x_{44}^{1})^{2}}} \frac{x_{44}^{1}}{\sqrt{1 - (x_{44}^{1})^{2}}} (A^{1})' + \frac{1}{\sqrt{1 - (x_{44}^{1})^{2}}} (E^{1})',$$

$$B^{3} = -\frac{x_{4}^{1}}{\sqrt{1 - (x_{44}^{1})^{2} - (x_{44}^{1})^{2}}} (E^{2})' + \frac{\sqrt{1 - (x_{44}^{1})^{2}}}{\sqrt{1 - (x_{44}^{1})^{2} - (x_{44}^{1})^{2}}} (B^{3})',$$

$$E^{2} = \frac{1}{\sqrt{1 - (x_{44}^{1})^{2} - (x_{44}^{1})^{2}}} (E^{2})' - \frac{x_{4}^{1}}{\sqrt{1 - (x_{44}^{1})^{2} - (x_{44}^{1})^{2}}} \frac{1}{\sqrt{1 - (x_{44}^{1})^{2}}} (B^{3})',$$

$$B^{2} = \frac{x_{4}^{1}}{\sqrt{1 - (x_{44}^{1})^{2} - (x_{44}^{1})^{2}}} (E^{3})' + \frac{\sqrt{1 - (x_{44}^{1})^{2}}}{\sqrt{1 - (x_{44}^{1})^{2} - (x_{44}^{1})^{2}}} (B^{2})',$$

$$E^{3} = \frac{1}{\sqrt{1 - (x_{44}^{1})^{2} - (x_{44}^{1})^{2}}}} (E^{3})' + \frac{x_{4}^{1}}{\sqrt{1 - (x_{44}^{1})^{2} - (x_{44}^{1})^{2}}} \frac{1}{\sqrt{1 - (x_{44}^{1})^{2}}} (B^{2})'.$$

Note that in these relations addends proportional to the derivatives of electromagnetic field tensor are omitted, and all variables are dimensionless. The intermixing of potential components with tensor components distinguishes essentially these transformations from the standard SRT transformations.

At zero velocity, we obtain

$$\varphi = \frac{1}{\sqrt{1 - (\boldsymbol{x}_{44}^{1})^{2}}} (\varphi)' + \frac{\boldsymbol{x}_{44}^{1}}{\sqrt{1 - (\boldsymbol{x}_{44}^{1})^{2}}} (E^{1})', \qquad \mathcal{A}^{1} = (\mathcal{A}^{1})',$$

$$E^{1} = \frac{\boldsymbol{x}_{44}^{1}}{\sqrt{1 - (\boldsymbol{x}_{44}^{1})^{2}}} (\varphi)' + \frac{1}{\sqrt{1 - (\boldsymbol{x}_{44}^{1})^{2}}} (E^{1})', \qquad B^{2} = (B^{2})', \qquad B^{3} = (B^{3})',$$

$$E^{2} = \frac{1}{\sqrt{1 - (\boldsymbol{x}_{44}^{1})^{2}}} (E^{2})', \qquad E^{3} = \frac{1}{\sqrt{1 - (\boldsymbol{x}_{44}^{1})^{2}}} (E^{3})'.$$

Let a point charge, e, be in accelerated motion with respect to an observer. In a proper frame of charge,

$$(\varphi)' = \frac{k_{\varphi} e}{r}, \qquad (\mathcal{A}^1)' = 0, \qquad (E)' = \frac{k_E e}{r^2}.$$

The coefficients k_{φ} and $k_{\rm E}$ are introduced here in order that the scalar potential and the electric intensity can be dimensionless. Therefore in the frame of observer at the direction of acceleration, electric intensity is determined by the following expression:

$$E^{1} = \frac{x_{44}^{1}}{\sqrt{1 - (x_{44}^{1})^{2}}} \frac{k_{\varphi} e}{r} + \frac{1}{\sqrt{1 - (x_{44}^{1})^{2}}} \frac{k_{E} e}{r^{2}}.$$

For $a \ll A$, electric intensity contains two addends, the first one is proportional to acceleration and varies inversely with distance to charge, while the second one does not depend from acceleration and is in inverse ratio with distance square. It is well known result which can be obtained by means of Lienart-Wiechert potentials (see, e.g., [6]). Thus the transformations derived allow, in particular, to determine field of accelerated charge without using Maxwell equations.

5. Relativistic mechanics

Relativistic mechanics generalized to accelerated motion can be constructed by analogy with relativistic mechanics being invariant with respect to uniform velocity motion.

5.1. 7-dimensional velocity

Differentials included in the expression for the interval square

$$(\mathrm{d}s)^2 = c^2 (\mathrm{d}t)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2 - T^2 (\mathrm{d}v^1)^2 - T^2 (\mathrm{d}v^2)^2 - T^2 (\mathrm{d}v^3)^2$$

can be considered as coordinates of vector in 7-space. We have

$$dx^{\alpha} = \{c dt, dx^{1}, dx^{2}, dx^{3}, T dv^{1}, T dv^{2}, T dv^{3}\}\$$

for contravariant coordinates of vector and

$$dx_{\alpha} = \{c dt, -dx_1, -dx_2, -dx_3, -T dv_1, -T dv_2, -T dv_3\}$$

for covariant coordinates of vector. Using the introduced coordinates we rewrite the interval square in form

$$(\mathrm{d}s)^2 = \mathrm{d}x^\alpha \, \mathrm{d}x_\alpha \,, \qquad (\alpha = 1, \dots, 7) \,.$$

Let us introduce a generalized Lorentz factor

$$\gamma = \left(1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}\right)^{-1/2} .$$

Expressing the interval as

$$ds = \frac{c \, dt}{\gamma}$$

we define 7-dimensional velocity as

$$u^{\alpha} \equiv \frac{\mathrm{d}x^{\mu}}{\mathrm{d}s} = \frac{\partial s}{\partial x_{\mu}} = \left\{ \gamma \,, \, \gamma \, \frac{v^b}{c} \,, \, \gamma \, \frac{a^b}{A} \right\}$$

in contravariant coordinates and

$$u_{\alpha} \equiv \frac{\mathrm{d}x_{\mu}}{\mathrm{d}s} = \frac{\partial s}{\partial x^{\mu}} = \left\{ \gamma \,, \, -\gamma \, \frac{v_b}{c} \,, \, -\gamma \, \frac{a_b}{A} \right\}$$

in covariant coordinates. Here b = 1, 2, 3, $v^b = v_b$, $a^b = a_b$, $v^b v_b = v^2$, $a^b a_b = a^2$. It is obvious that

$$u^{\alpha} u_{\alpha} = 1. (23)$$

5.2. Free particle action

We define a free particle action as

$$S = -m c \int_{s_1}^{s_2} \mathrm{d}s \,,$$

where the integration is over a line in 7-space, s_1 , s_2 are points of the specified line, and m is the particle mass. The action can be expressed as an integral over time

$$S = -m c^2 \int_{t_1}^{t_2} \sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}} \, dt \,.$$

5.3. Energy, impulse, force

We define a 7-dimensional impulse as

$$p^{\alpha} = -\frac{\partial S}{\partial x_{\alpha}} = m c u^{\alpha}$$
 and $p_{\alpha} = -\frac{\partial S}{\partial x^{\alpha}} = m c u_{\alpha}$

in contravariant and covariant coordinates, respectively. Let us introduce a relativistic energy

$$E = \gamma \, m \, c^2 \,,$$

a relativistic impulse

$$p = \gamma m v$$
,

and a relativistic kinetic force

$$f = \gamma m a$$
.

Using the introduced quantities we can express the components of 7-impulse in form

$$p^{\alpha} = \left\{ \frac{E}{c}, p^b, T f^b \right\}$$
 and $p_{\alpha} = \left\{ \frac{E}{c}, -p_b, -T f_b \right\}$.

From (23) follows

$$p^{\alpha} p_{\alpha} = m^2 c^2.$$

This can be written as the relation for energy, impulse, kinetic force, and mass in relativistic mechanics generalized to accelerated motions:

$$\frac{E^2}{c^2} - p^2 - T^2 f^2 = m^2 c^2. (24)$$

From it follows that massless accelerated particle is like to particle in uniform velocity motion with "effective" mass

$$\frac{1}{c}\sqrt{\frac{E^2}{c^2}-p^2} = \frac{f}{A}.$$

The transformation of 7-impulse components can be described by the formalism similar to that for the transformation of coordinate differentials. For example, in the case of **A** V-motion of frame, the transformations (20) imply

$$E = \frac{1}{\sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}}} (E)' + \frac{v}{\sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}}} \frac{1}{\sqrt{1 - \frac{a^2}{A^2}}} (p^1)' + \frac{a T^2}{\sqrt{1 - \frac{a^2}{A^2}}} (f^1)',$$

$$p^1 = \frac{v}{c^2 \sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}}} (E)' + \frac{\sqrt{1 - \frac{a^2}{A^2}}}{\sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}}} (p^1)',$$

$$f^1 = \frac{a}{c^2 \sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}}} (E)' + \frac{v a}{c^2 \sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}}} (f^1)' + \frac{1}{\sqrt{1 - \frac{a^2}{A^2}}} (f^1)'.$$

Unlike the usual SRT transformations of energy and impulse, the inclusion of accelerated motion leads to intermixing energy, components of impulse and of force.

5.4. Wave equation

7-dimensional derivative operators are given by

$$\frac{\partial}{\partial x_{\alpha}} = \left\{ \frac{1}{c} \frac{\partial}{\partial t}, -\frac{\partial}{\partial x_{b}}, -\frac{1}{T} \frac{\partial}{\partial v_{b}} \right\} \quad \text{and} \quad \frac{\partial}{\partial x^{\alpha}} = \left\{ \frac{1}{c} \frac{\partial}{\partial t}, \frac{\partial}{\partial x^{b}}, \frac{1}{T} \frac{\partial}{\partial v^{b}} \right\}.$$

Using these operators we get wave equation

$$\frac{\partial^2}{\partial x^\alpha \, \partial x_\alpha} \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{1}{T^2} \frac{\partial^2}{\partial v^2} = 0.$$

This wave equation is different from the traditional one by the last addend allowing to describe accelerated motion of wave as well as the processes of wave initiation and disappearance.

Let $\phi(t, x, v)$ be an arbitrary function describing the wave field. We shall try for a solution of wave equation in form

$$\phi(t, x, v) = \phi_0 \exp\left[i\left(\kappa_b x^b + \xi_b v^b - \omega t\right)\right], \qquad (25)$$

where ω is the wave frequency, κ_b are the coordinates of wave vector. By analogy ξ_b will be named coordinates of a wave vector of velocity.

Substitution of the expression (25) in the wave equation gives

$$\frac{\omega^2}{c^2} - \kappa^2 - \frac{\xi^2}{T^2} = 0. {26}$$

If we multiply this equation by the Planck constant square and compare the result to (24) within the framework of wave-corpuscle duality, we obtain a set of relations:

$$E = \hbar \, \omega \,, \qquad p = \hbar \, \kappa \,, \qquad f = \frac{\hbar}{T^2} \, \xi \,.$$

The first two are de Broglie relations, while the last one connects the wave vector of velocity with the relativistic kinetic force.

By analogy with the traditional definition of wave velocity

$$v^b \equiv \frac{\partial \omega}{\partial \kappa_b} = \frac{\kappa^b}{\omega} c^2 \,.$$

we define a wave acceleration

$$a^b \equiv \frac{\partial \omega}{\partial \xi_b} = \frac{\xi^b}{\omega} A^2 .$$

Substituting these expressions in (26) we find a relation

$$c^2 - v^2 - a^2 T^2 = 0, (27)$$

which will be named an *equation of wave motion*. If wave propagates in one direction, integration of the equation of motion (27) reduces to two solutions for wave velocity:

$$v_I = \pm c,$$

$$v_{II} = \pm c \cos(t/T + C_1),$$

where C_1 is integration constant. Thus two types of wave motion are possible. In the first case, wave is in linear motion at constant velocity c, and in the second case, wave oscillates along line at amplitude L. At time moments $t/T + C_1 = \pi n$, where n is integer, wave velocity is equal $\pm c$ for oscillatory motion, and the change of motion type is made possible. Then the oscillatory motion of wave can be transformed into the linear uniform velocity motion and vice versa. For two-dimensional case, the uniform circular motion is a particular solution of the equation (27). It is described relations:

for circle radius

$$r = \frac{L}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{v^2}{c^2} \,,$$

for centripetal acceleration

$$a = A\sqrt{1 - \frac{v^2}{c^2}}.$$

The wave equation generalizing the Klein-Gordon equation for massive particles to accelerated motion can be written in the following form

$$\frac{\partial^2 \psi}{\partial x^\alpha \, \partial x_\alpha} + \frac{m^2 \, c^2}{\hbar^2} \psi = 0 \, .$$

Substitution of (25) in the wave equation gives

$$\frac{\omega^2}{c^2} - \kappa^2 - \frac{\xi^2}{T^2} = \frac{m^2 c^2}{\hbar^2} \,. \tag{28}$$

Let's assume

$$\omega = \frac{m c^2}{\hbar} + \omega' \,,$$

where $\omega' \ll m c^2/\hbar$. For non-relativistic frequency ω' the equation (28) is reduced to following

$$\frac{2\,m\,\omega'}{\hbar} - \kappa^2 - \frac{\xi^2}{T^2} = 0\,. \tag{29}$$

By analogy to the traditional definition of wave packet group velocity

$$v_{\rm g}^b \equiv \frac{\partial \omega'}{\partial \kappa_b} = \frac{\hbar}{m} \, \kappa^b$$

we define a wave packet group acceleration

$$a_{\rm g}^b \equiv \frac{\partial \omega'}{\partial \xi_b} = \frac{\hbar}{m} \frac{\xi^b}{T^2} \,.$$

By using group velocity and acceleration, (29) is reduced to the expression

$$\hbar\,\omega' = \frac{m\,v_{\rm g}^2}{2} + \frac{m\,T^2\,a_{\rm g}^2}{2}\,,$$

which is a generalization of known de Broglie's relation.

6. Conclusions

In the present paper we have developed the self-consistent SRT generalization to accelerated motions which is founded only on the principle of the uniform description of kinematic parameters. This principle is, as applied to SRT, in the existence of relations between derivations and turn angles in the space of kinematic variables.

In spite of formal character of initial positions, the generalized SRT gives farreaching physical consequences. In particular, the new fundamental constant T of time dimensionality should be introduced. This constant breeds a set of secondary constants: the fundamental length L = cT, and the maximal acceleration A = c/T. Unfortunately, the experimental verification of the relativistic composition rule for accelerations seems to be as yet impossible because of the extraordinarily high value of the maximal acceleration (for example, Scarpetta estimated it at 5×10^{53} cm/s²).

We study accelerated motions in the context differed radically from Caianiello's conception (see [7] and references therein). However our results are similar to those obtained in works of Caianiello and his colleagues [5, 8]. By methodical reasons, we expand space-time through three coordinates of velocity instead of coordinates of four-velocity used in Caianiello's model. This explains the difference between coordinate transformations, and between relations for kinematic and dynamic parameters in our and Scarpetta's considerations.

Since in according to the SRT, three velocity components and three space turn angles make full Lorentz group, the present work can be naturally generalized to uniform rotations of reference frames. An universal "relativistic" approach to any non-inertial motions and, in particular, to uniform rotations will be given in a forthcoming paper [9].

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